IX.—*On the Determination of the Intensity of the Earth's Magnetic Force in absolute Measure, by means of the Dip-Circle.* *By* Humphrey Lloyd, *D.D., D.C.L., F.R.SS.L. & E.; V.P.R.I.A; Corresponding Member of the Royal Society of Sciences at Gottingen; Honorary Member of the American Philosophical Society, of the Batavian Society of Sciences, and of the Societè de Physique et d'Histoire Naturelle of Geneva, &c. &c.*

Read January 11, 1858.

The force exerted by the Earth upon a magnet is usually found by suspending the bar horizontally, and observing its time of vibration. The result thence obtained is the product of the horizontal component of the Earth's magnetic force by the magnetic moment of the magnet; and before we can know the value of either of the factors which compose it, observation must furnish another result in which they are combined differently. This is effected, in the method of Gauss, by using the same magnet to deflect another, similarly suspended, and by observing the angles of deflection at known distances: this observation gives the ratio of the magnetic moment of the deflecting magnet to the horizontal component of the Earth's force, and the two factors are therefore absolutely known.

This method, although much improved by the labours of Lamont and others, has one insurmountable imperfection. The total force must be inferred from its horizontal component, by multiplying by the secant of the inclination; the relative error of the deduced force, arising from a given error of inclination, varies, therefore, as the tangent of that angle, and when the inclination approaches to 90° it becomes very considerable. The method is, accordingly, unsuited to the high magnetic latitudes.

I was induced to consider the means of supplying this defect some years ago, upon the occasion of the Arctic Expeditions of 1845 and 1848; and I then suggested another process, by which the total intensity might be found directly, without the intervention of its horizontal component. In the paper in which it was explained,* it was shown that the ordinary dip-circle may be employed in the two parts of the observation,—the product of the Earth's magnetic force into the moment of free magnetism of the needle being determined, by observing the position of equilibrium of the dipping-needle loaded with a small weight, and the ratio of the same quantities being found (as in GAUSS's method) by removing the needle, and employing it to deflect another substituted in its place. Subsequent considerations, however, derived from the probable errors of observation, led me to propose that the dip-circle should be employed only in one part of this process, and that the observation should be completed by the known methods.

The present communication is intended to show in what manner this complication may be avoided, and the original proposal carried out. It is of great importance to the scientific traveller that the instruments which he has to carry should be reduced, as far as possible, in number and in weight, and that their adjustments should be few and simple; and it is believed that these objects are attainable by the method explained in this paper. Before entering into details it will be convenient to revert to the simple theoretical principles on which it is founded.

If X and Y denote the horizontal and vertical components of the Earth's magnetic force, M the magnetic moment of the needle acted on, and α the azimuth of the plane in which it moves, measured from the magnetic meridian, the effective forces exerted upon it are—

$$MX \cos \alpha, \quad MY;$$

and their moment to turn the needle is—

$$M(Y \cos \eta - X \cos \alpha \sin \eta),$$

η denoting the actual inclination of the needle to the horizon. This moment is opposed by that of the added weight, or by Wr, W being the weight, and r

* Proceedings of the Royal Irish Academy, January 24, 1848.

the radius of the pulley by which it acts;* and the equation of equilibrium is therefore—

$$M(Y\cos\eta - X\cos\alpha\sin\eta) = Wr. \qquad (1)$$

When the needle is removed, in the second part of the process, and applied to deflect another substituted in its place, the moment of its force to turn the latter is

$$MM'U;$$

in which M' is the moment of free magnetism of the second needle, and U a function of the distance of the centres of the two needles, and of certain integrals depending on the distribution of free magnetism in them. The moment of the Earth's magnetic force, opposed to this, is of the form already assigned, in which we have only to substitute M', η', and α', for M, η, and α. Hence the equation of equilibrium is

$$Y\cos\eta' - X\cos\alpha'\sin\eta' = MU; \qquad (2)$$

the quantity M' disappearing from the result. The magnetic moment of the deflecting needle, M, is eliminated from equations (1) and (2) by multiplication; and we shall thus obtain a single relation between the intensity of the Earth's magnetic force, the observed angles α, η, α', η', and the quantities W, r, and U. Hence the magnetic intensity will be determined when these are known.

There are three obvious cases of these formulæ, each of which suggests a different method for the determination of the terrestrial magnetic intensity.

1. When the planes in which the needles move coincide with the magnetic meridian, or $\alpha = 0$, $\alpha' = 0$, the left-hand members of (1) and (2) are reduced to $MR\sin(\theta - \eta)$, $R\sin(\theta - \eta')$; R denoting the total force, and θ the inclination. Whence, by multiplication, we have

$$R^2\sin(\theta - \eta)\sin(\theta - \eta') = UWr. \qquad (3)$$

* It is here supposed that the weight is attached to a fine thread passing round a light pulley, whose centre is on the axis of the cylindrical axle of the needle, in the manner proposed by Mr. Fox. If the weight be attached to the southern arm of the needle, at a fixed point, its moment is $Wr\cos\eta$.

2. When the planes in which the needles move are perpendicular to the magnetic meridian, or $\alpha=90°$, $\alpha'=90°$, the left-hand members of (1) and (2) become, respectively, $MY\cos\eta$, $Y\cos\eta'$; whence

$$Y^2\cos\eta\cos\eta'=UWr. \tag{4}$$

3. Finally, the equilibrium may be produced, in both cases, by turning the instrument in azimuth until the free needle stands vertically. In this case $\eta=90°$, $\eta'=90°$, and the left-hand members become $-MX\cos\alpha$, $-X\cos\alpha'$; whence

$$X^2\cos\alpha\cos\alpha'=UWr. \tag{5}$$

Thus we may apply this principle to the determination of the total intensity, or to that of either of its two components.

In comparing the foregoing methods, it is to be observed that the third fails when the inclination approaches to 90°, on account of the magnitude of the error of R resulting from a given error of θ, when the total force is deduced from its horizontal component. In like manner, and for the same reason, the second method fails in the vicinity of the magnetic equator, or line of no inclination. The first alone is applicable at all parts of the Earth's surface, and I proceed to consider it more in detail.

The observed angles, η and η', are liable to error, the friction of the needles on their supports causing them to rest in positions slightly different from those due to the acting forces. The probable errors of η and η', due to this cause, vary with the angles themselves. To determine their magnitude in any case, we have

$$MR\sin(\theta-\eta)=F,$$

F being the moment of the deflecting force; and when friction is taken into account,

$$MR\sin(\theta-\eta+\Delta\eta)=F+f;$$

f denoting the moment of friction, and $\eta-\Delta\eta$ the new angle of equilibrium. Developing the latter equation, and subtracting the former,

$$MR\cos(\theta-\eta)\,\Delta\eta=f;$$

the angle $\Delta\eta$ being expressed in parts of radius. Hence, $\cos(\theta-\eta)\Delta\eta$ is constant with a given instrument, and at a given point of the Earth's surface.

To find the probable error of the force corresponding to the error of the observed angle, we must differentiate the equation of equilibrium, $MR \sin u = F$, with respect to R and u, where $u = \theta - \eta$; and we have

$$\Delta R \sin u + R \cos u \, \Delta u = 0.$$

But

$$u = \tfrac{1}{2} (\eta_1 - \eta_2),$$

η_1 and η_2 being the observed angles of inclination under the two opposite actions of the deflecting force. Hence, the *probable error* of u is

$$\Delta u = \tfrac{1}{2} \sqrt{\Delta\eta^2_1 + \Delta\eta^2_2} = \frac{1}{\sqrt{2}} \Delta\eta;$$

since $\Delta\eta_2 = \Delta\eta_1$. Accordingly, the second term of the preceding equation becomes $\dfrac{1}{\sqrt{2}} R \cos u \, \Delta\eta = \dfrac{1}{\sqrt{2}} \dfrac{f}{M}$; and we have

$$\Delta R = \frac{-f}{M \sqrt{2} \sin u}.$$

We learn, then, that the probable error of the force varies inversely as the sine of the angle of deflection; and that it is therefore requisite for accuracy that this angle should be considerable. There is no difficulty in augmenting the angle of deflection as much as we please in the first part of the process, in which the magnet is deflected by a weight; but in the second, the case is different, and with the slender needles to be employed as deflectors, a large deflection can only be obtained by placing the deflecting needle at a very short distance from the moveable one. The most convenient arrangement appears to be to attach the deflecting needle to the moveable arm of the divided circle which carries the verniers, and at right angles to the wires of the microscopes.* So attached, it must always be rendered perpendicular to the

* To obtain the value of ΔR by observation, we must substitute for f its value given above. But when $\eta = \theta$, or when the needle is undeflected, $f = MR\Delta\theta$; wherefore

$$\Delta R = \frac{-R\Delta\theta}{\sqrt{2} \sin u}.$$

In the instrument with which I made trial of this method, the length of the needles was $3\frac{1}{2}$ inches, and the angle of deflection produced, in the position of the deflecting needle here described, was

deflected needle in the course of the observation, although in a different plane.

The form of the function denoted by U, in this position, is easily found.

Let the distances of any points of the axes of the deflecting and deflected magnets from their respective centres be denoted by r and r', and let μ and μ' denote the quantities of free magnetism at these points, contained in the slices perpendicular to the axes whose thicknesses are dr and dr'. Then the force exerted by the former upon the latter is

$$\frac{\mu\mu' dr\, dr'}{z^2},$$

z denoting their mutual distance. The portion of this force contained in the plane of the deflected magnet, and perpendicular to its axis, is

$$\frac{\mu\mu' dr\, dr'}{z^2} \times \frac{r}{z};$$

and the moment of this force to turn the magnet is obtained by multiplying by r'. But

$$z^2 = D^2 + r^2 + r'^2,$$

D being the distance of the centres of the two magnets; and accordingly the total moment of the acting forces is

$$\iint \frac{\mu r dr \,.\, \mu' r' dr'}{(D^2 + r^2 + r'^2)^{\frac{3}{2}}}.$$

Expanding the denominator, and making, for abridgment,

$$\begin{array}{llll} \int \mu r dr = M_1, & \int \mu r^3 dr = M_3, & \int \mu r^5 dr = M_5, & \&c., \\ \int \mu' r' dr' = M'_1, & \int \mu' r'^3 dr' = M'_3, & \int \mu' r'^5 dr' = M'_5, & \&c., \end{array}$$

in which the integrals are to be taken between the limits $r = \pm l$, $r' = \pm l'$, l and l' being half the lengths of the two magnets, this becomes

24° 10'. But the probable error of a *single reading* of the inclination, obtained by repetition—the needle being lifted off the agate planes between the successive readings—was 1'·6; and if *four* readings (which is a very usual number) be taken, the probable error of their *mean* will be one-half of this. In this case, therefore, $\Delta\theta = 0'{\cdot}8$; and the probable error of the deduced force, computed by the preceding formula, is $\Delta R = {\cdot}00040\, R$.

$$\frac{1}{D^3}\left\{MM' - \frac{3}{2}(M_3M' + MM'_3)\frac{1}{D^2} + \frac{3.5}{2.4}(M_5M' + 2M_3M'_3 + MM'_5)\frac{1}{D^4} + \&c.\right\},$$

or $MM'U$, in which

$$U = \frac{1}{D^3}\left\{1 - \frac{3}{2}\left(\frac{M_3}{M} + \frac{M'_3}{M'}\right)\frac{1}{D^2} + \frac{15}{8}\left(\frac{M_5}{M} + 2\frac{M_3}{M}\frac{M'_3}{M'} + \frac{M'_5}{M'}\right)\frac{1}{D^4} + \&c.\right\}$$

Now it is to be observed, that the variations of the ratios $\frac{M_3}{M}, \frac{M_5}{M}$, &c., arising from the variations of μ, are of a lower order of magnitude than that of M, and may be disregarded in their effect upon the value of U.* On the supposition that the quantity of free magnetism, at any point of a magnet, is proportional to the distance from the centre, or that $\mu = kr$, we have

$$M = \tfrac{2}{3}kl^3, \qquad M_3 = \tfrac{2}{5}kl^5, \qquad M_5 = \tfrac{2}{7}kl^7,$$

and when k becomes $k - \delta k$, these values will all be altered proportionally, and consequently the ratios $\frac{M_3}{M}, \frac{M_5}{M}$, &c., will be absolutely unchanged; and the same thing is manifestly true, if the quantity of free magnetism be supposed to vary as any simple power of the distance, whether integer or fractional.

This is a point of considerable importance in reference to the method now proposed. For it follows that, at a given distance between the two needles, the function U may be regarded as *constant;* and, therefore, that, even when U is unknown, the value of R will be relatively determined, by a process which is independent of the changes induced by time in the magnetic moments of the needles employed. Accordingly, if the value of the force be found at *any one place*, by any independent means, it will be absolutely known at all; and it is only necessary that the observer should include in his series an observation at some base-station, at which the absolute value of the force is determined simultaneously by the ordinary method.

I now proceed to show, however, that the value of the constant U may be found by deflection, by the instrument itself, and without any subsidiary apparatus; and that the method may therefore be rendered rigorously *absolute*. It is obvious that the ordinary process is inapplicable in this case, owing to the large number of terms which acquire a sensible value, in the value of the func-

* This circumstance was first pointed out by Dr. LAMONT.

tion U, and the consequent difficulty and uncertainty of the elimination: moreover, the position which has been adopted for the deflecting needle will not admit of the required alteration of distance.

Now here I premise, that it is not necessary that the usual deflection distance should be one of the series employed in deducing the coefficients of the inverse powers of the distance in the value of U: it is not even requisite that the relative positions of the two magnets should be the same in the two cases. For if the value of the corresponding function be found, for *any other position*, and at *any distance*, that of U will be known by a comparison of the deflections produced. Accordingly, I propose to determine, in the first place, the value of the corresponding function in a different relative position of the two magnets, and by means of deflections at the usual distances; and thence to conclude that of U in the position of the magnets here employed.

In using the dip-circle for this purpose, it will be found most convenient to adopt the third of the methods above described, in which the equilibrium is produced by turning the instrument in azimuth until the deflected magnet becomes vertical; for in this case the deflecting magnet is always horizontal, and can be placed in the usual position with respect to the deflected magnet without difficulty. For this purpose the apparatus is provided with a gun-metal bar, the middle of which is broad, and has a rectangular aperture which enables it to pass over the box containing the deflected magnet: this bar rests horizontally on two supports fixed outside the box, on the level of the agate planes. The deflecting magnet is to be placed on this support at different known distances, and on each side of the deflected magnet, its axis being in the plane in which the latter moves; and the apparatus is to be turned in azimuth until the deflected needle is vertical. In this case equation (2) becomes

$$-X \cos a = MV;$$

in which V is of the form

$$V = \frac{2}{D^3}\left\{1 + \frac{p}{D^2} + \frac{q}{D^4} + \&c.\right\},$$

where

$$p = 2\frac{M_3}{M} - 3\frac{M'_3}{M'}, \qquad q = 3\frac{M_5}{M} - 15\frac{M_3}{M}\frac{M'_3}{M'} + \frac{45}{8}\frac{M'_5}{M'}.$$

Let V_1, V_2, V_3, &c., denote the values of V corresponding to the distances D_1,

D_2, D_3, &c.; and a_1, a_2, a_3, &c., the corresponding azimuths observed; then it is obvious that

$$V_2 \cos a_1 - V_1 \cos a_2 = 0, \qquad V_3 \cos a_1 - V_1 \cos a_3 = 0, \text{ \&c.,}$$

from which equations the values of the coefficients p, q, &c., are obtained by elimination in the usual manner. Hence V is completely determined.

Now let the deflecting magnet be removed from the horizontal bar, and placed in its ordinary position between the microscopes; and let the observation be repeated, the instrument being turned in azimuth until the deflected needle is vertical. Then, if a_0 denote the corresponding azimuth, we have

$$-X \cos a_0 = MU;$$

whence there is

$$U = V \frac{\cos a_0}{\cos a}.$$

Thus the uncertainty of the result, arising from the smallness of the angle of deflection at the usual distances, is removed from the regular series of observations, and is thrown upon the determination of the constant, which may be made at leisure, whenever convenient, and may be repeated as often as is required for accuracy.

In speaking of U as constant, and independent of the changes of the magnetic moments of the needles, I have expressly limited the statement to those small and regular losses of magnetism which occur in time. It would not be safe to extend the assumption to the case of the larger changes brought about abruptly by concussion, or any other accidental cause; and still less to those in which the magnetic distribution of the needles is altered by contact with, or proximity to, magnetic bodies. In such cases (the occurrence of which is easily detected) the value of the constant U should be re-determined.

It may be useful to add a few words respecting the order of the observations.

The apparatus should be furnished with three needles, all of the same size, viz., $3\frac{1}{2}$ inches in length. One of these (which we shall denote by the letter A) is to be employed in observations of inclination: the other two, B and C, are to be used in the observations of intensity,—B being the loaded needle, which is

also used as a deflector, and C the deflected needle. The two latter needles should not have their poles reversed, nor be touched with a magnet.

The observations may be conveniently taken in the following order:—

1. Needle A is to be placed on the agate planes, the other two needles being removed, and a complete observation of inclination taken in the four usual positions of the needle and limb, and with the poles of the needle direct and inverted.

2. Needle A is then to be removed, and the loaded needle, B, substituted; and its inclination to the horizon, η, is to be observed in the four positions of the needle and limb. The deviation of this needle from the position due to the Earth's magnetic force alone is $u=\theta-\eta$, the angle η being *positive* when measured at the same side of the horizontal line with θ, and negative in the contrary case.

3. Needle B is now to be removed from the agate planes to its supports between the microscopes, and needle C substituted; and the inclination of the latter to the horizon is then to be observed in one position of the needle and limb. The observation is to be repeated with the north end of needle B turned in the opposite direction, by the revolution of the moveable arms which carry the microscopes; half the difference of the readings in the two positions is the angle of deflection, u'.

The total intensity is given by the formula

$$R=\sqrt{\frac{UWr}{\sin u \sin u'}},$$

which is fitted, without any artifice, for logarithmic computation.*

In strictness a correction is required for the effect of the change of temperature of needle B, in the two observations in which it is employed; but as one of these observations may be made to follow the other quickly, and as the needle may be placed in both in nearly the same circumstances, the correction may generally be disregarded. It will be necessary, however, that this needle,

* If the weight be attached to the loaded needle at a fixed point, the formula becomes

$$R=\sqrt{\frac{UWr\,.\cos\eta}{\sin u \sin u'}}.$$

when employed as a deflector, should be protected from the heat of the observer's body by a small case of glass or of metal.

The method here proposed appears to offer the following advantages to the travelling observer :—

1. It is applicable, with equal accuracy, at all parts of the globe.

2. It dispenses with the employment of a separate instrument for the determination of the magnetic intensity, and with the separate adjustments required in erecting it.

3. The constants to be determined—the magnitude of the added weight, and the radius of the pulley by which it acts—can be ascertained with more ease and certainty than those which are required in the method of vibrations, and are less liable to subsequent change.

4. The observations themselves are less varied in character than the usual ones, and may be completed in a shorter time.